$$y = \cos(\omega t + \phi)\left[A \cosh \frac{\omega x}{v} + B \sinh \frac{\omega x}{v} + C \cos \frac{\omega x}{v} + D \sin \frac{\omega x}{v}\right]$$

Free - Free

$$\frac{\partial^2 y}{\partial x^2} = 0 \qquad \frac{\partial^3 y}{\partial x^3} = 0$$

$$y'' = \left[A\left(\frac{\omega}{v}\right)^2 \cosh \frac{\omega x}{v} + B\left(\frac{\omega}{v}\right)^2 \sinh \frac{\omega x}{v} - C\left(\frac{\omega}{v}\right)^2 \cos \frac{\omega x}{v} - D\left(\frac{\omega}{v}\right)^2 \sin \frac{\omega x}{v}\right]$$

$$y''' = \left[A\left(\frac{\omega}{v}\right)^3 \sinh \frac{\omega x}{v} + B\left(\frac{\omega}{v}\right)^3 \cosh \frac{\omega x}{v} + C\left(\frac{\omega}{v}\right)^3 \sin\left(\frac{\omega x}{v}\right) - D\left(\frac{\omega}{v}\right)^2 \cos \frac{\omega x}{v}\right]$$

At $x = 0$

$x=0$ $y'' = 0 = A \cosh 0 + B \sinh 0 - C \cos 0 - D \sin 0$

$x=0$ $y''' = 0 = A \sinh 0 + B \cosh 0 + C \sin 0 - D \cos 0$

$x=L$ $y'' = 0 = A \cosh \frac{\omega L}{v} + B \sinh \frac{\omega L}{v} - C \cos \frac{\omega L}{v} - D \sin \frac{\omega L}{v}$

$x=L$ $y''' = 0 = A \sinh \frac{\omega L}{v} + B \cosh \frac{\omega L}{v} + C \sin \frac{\omega L}{v} - D \cos \frac{\omega L}{v}$

$0 = A - C \qquad \therefore A = C$

$0 = B - D \qquad \therefore B = D$

let $\gamma = \frac{\omega L}{v}$

$0 = A \cosh \gamma + B \sinh \gamma - A \cos \gamma - B \sin \gamma$

$0 = A \sinh \gamma + B \cosh \gamma + A \sin \gamma - B \cos \gamma$

$$0 = A(\cosh \gamma - \cos \gamma) + B(\sinh \gamma - \sin \gamma)$$

$$0 = A(\sinh \gamma + \sin \gamma) + B(\cosh \gamma - \cos \gamma)$$

$$A(\cosh\gamma - \cos\gamma) = -B(\sinh\gamma - \sin\gamma)$$
$$A(\sinh\gamma + \sin\gamma) = -B(\cosh\gamma - \cos\gamma)$$

$$\frac{(\cosh\gamma - \cos\gamma)}{(\sinh\gamma + \sin\gamma)} = \frac{(\sinh\gamma - \sin\gamma)}{(\cosh\gamma - \cos\gamma)}$$

$$(ch\gamma - c\gamma)(ch\gamma - c\gamma) = (sh\gamma - s\gamma)(sh\gamma + s\gamma)$$

$$ch^2\gamma + c^2\gamma - 2\,c\gamma\, ch\gamma = sh^2\gamma - s^2\gamma$$

$$ch^2\gamma - sh^2\gamma - 2ch\gamma c\gamma = -(c^2\gamma + s^2\gamma)$$

$$\cancel{ch^2\gamma - sh^2\gamma} + 1 - 2ch\gamma c\gamma = -1$$

$$-2ch\gamma c\gamma = -2$$

$$1 - 2ch\gamma c\gamma = -1 \Rightarrow \boxed{ch\gamma c\gamma = 1}$$

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look at <u>Fixed-Free</u>

$$\frac{ch\gamma + c\gamma}{sh\gamma - s\gamma} = \frac{sh\gamma + s\gamma}{ch\gamma + c\gamma}$$

$$(ch\gamma + c\gamma)^2 = (sh\gamma + s\gamma)(sh\gamma - s\gamma)$$

$$ch^2\gamma + c^2\gamma + 2ch\gamma c\gamma = sh^2\gamma - s^2\gamma$$

$$ch^2\gamma - sh^2\gamma + 2ch\gamma c\gamma = -c^2\gamma - s^2\gamma$$

$$1 + 2ch\gamma c\gamma = -1$$

$$2ch\gamma c\gamma = -2$$

$$\boxed{ch\gamma c\gamma = -1}$$

#3 $\cosh\gamma \; \cos\gamma = 1$

#1 $\tan\gamma/2 = \pm\sqrt{\dfrac{1-\cos\gamma}{1+\cos\gamma}}$

$\cos\gamma = \dfrac{1}{\cosh\gamma}$

#2 $\tanh\gamma/2 = \pm\sqrt{\dfrac{\cosh\gamma - 1}{\cosh\gamma + 1}}$

sub into #1

$$\tan\gamma/2 = \pm\sqrt{\dfrac{1-\frac{1}{\cosh\gamma}}{1+\frac{1}{\cosh\gamma}}}$$

$$\tan\gamma/2 = \pm\sqrt{\dfrac{\frac{\cosh\gamma-1}{\cosh\gamma}}{\frac{\cosh\gamma+1}{\cosh\gamma}}} = \pm\sqrt{\dfrac{\cosh\gamma-1}{\cosh\gamma+1}}$$

$$\therefore \tan\gamma/2 = \pm\tanh\gamma/2 \longrightarrow \boxed{\cancel{\tan\gamma/2 \coth\gamma/2 = 1}}$$

$\cosh\gamma \cos\gamma = -1$

$$\cos\gamma = \frac{-1}{\cosh\gamma}$$

$$\tan\gamma/2 = \pm\sqrt{\dfrac{1-\frac{-1}{\cosh\gamma}}{1+\frac{-1}{\cosh\gamma}}}$$

$$= \pm\sqrt{\dfrac{\frac{\cosh\gamma+1}{\cosh\gamma}}{\frac{\cosh\gamma-1}{\cosh\gamma}}} = \pm\sqrt{\dfrac{\cosh\gamma+1}{\cosh\gamma-1}}$$

$$\tan\gamma/2 = \frac{1}{\pm\sqrt{\dfrac{\cosh\gamma-1}{\cosh\gamma+1}}} = \frac{1}{\pm\tanh\gamma/2}$$

$$\tan\gamma/2 = \pm\coth\gamma/2$$

$$\text{or } \cot\gamma/2 = \pm\tanh\gamma/2$$