Chapter 35 Problem 49 †

Given

$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U(x,y,z)\psi=E\psi$

Solution

a) Show that $\psi = A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L)$ is a solution to Schrodinger's equation for a 3-dimensional well.

Inside the 3-dimensional well the potential energy function is zero.

$$U(x,y,z)=0$$

Take the second partial derivative of the function with respect to x, y and z.

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial^2}{\partial x^2}(A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L))$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{-n_x^2\pi^2}{L^2}(A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L))=\frac{-n_x^2\pi^2}{L^2}\psi$$

$$\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2}{\partial y^2}(A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L))$$

$$\frac{\partial^2\psi}{\partial y^2}=\frac{-n_y^2\pi^2}{L^2}(A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L))=\frac{-n_y^2\pi^2}{L^2}\psi$$

$$\frac{\partial^2\psi}{\partial z^2}=\frac{\partial^2}{\partial z^2}(A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L))$$

$$\frac{\partial^2\psi}{\partial z^2}=\frac{-n_z^2\pi^2}{L^2}(A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L))=\frac{-n_z^2\pi^2}{L^2}\psi$$

Then the left side of Schrodinger's equation becomes

$$-\frac{\hbar^2}{2m}\left(\frac{-n_x^2\pi^2}{L^2}\psi+\frac{-n_y^2\pi^2}{L^2}\psi+\frac{-n_z^2\pi^2}{L^2}\psi\right)=\frac{\hbar^2}{2m}\left(\frac{n_x^2\pi^2}{L^2}+\frac{n_y^2\pi^2}{L^2}+\frac{n_z^2\pi^2}{L^2}\right)\psi$$

Comparing this to the right side of the equation, then Schrodinger's equation is satisfied as long as

$$E=\frac{\hbar^2}{2m}\left(\frac{n_x^2\pi^2}{L^2}+\frac{n_y^2\pi^2}{L^2}+\frac{n_z^2\pi^2}{L^2}\right)$$

b) Verify that the energy relationship matches that of the textbook.

Remember

$$\hbar=\frac{h}{2\pi}$$

Then

$$E=\frac{(h/2\pi)^2}{2m}\left(\frac{n_x^2\pi^2}{L^2}+\frac{n_y^2\pi^2}{L^2}+\frac{n_z^2\pi^2}{L^2}\right)=\frac{h^2}{8m\pi^2}\left(\frac{n_x^2\pi^2}{L^2}+\frac{n_y^2\pi^2}{L^2}+\frac{n_z^2\pi^2}{L^2}\right)$$

$$E=\frac{h^2}{8m}\left(\frac{n_x^2}{L^2}+\frac{n_y^2}{L^2}+\frac{n_z^2}{L^2}\right)=\frac{h^2}{8mL^2}\left(n_x^2+n_y^2+n_z^2\right)$$

† Problem from Essential University Physics, Wolfson