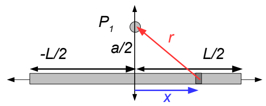
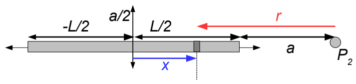
Chapter 5 Problem 87 †





Solution

The charge is uniformly distributed over the rod. Therefore, $\lambda = q/L$.

The electric field for a continuous linear charge on this wire is

$$\vec{E} = \int_{x=-L/2}^{x=L/2} \left(\frac{kdq}{r^2} \hat{r} \right)$$

The infinitesimal charge is related to the infinitesimal length along the charged wire by the following relationship.

$$dq = \lambda dx = \frac{qdx}{L}$$

Depending on the position of our point of interest, how we define r and $\hat{r}$ will change. Eventually everything needs to be defined in terms of the variable of integration, x.

$$\vec{E} = \int_{x=-L/2}^{x=L/2} \left(\frac{k\lambda dx}{r^2} \hat{r} \right) = k\lambda \int_{x=-L/2}^{x=L/2} \left(\frac{dx}{r^2} \hat{r} \right)$$

Now we can look at the two cases.

What is the electric field at P_1?

In this case, the vector r goes from a location on the wire a distance, x, from the origin to a point that is $a/2$ above the middle of the wire in the x-direction. Therefore,

$$\vec{r} = -x\hat{i} + a/2\hat{j}$$

The magnitude is

$$r = \sqrt{(-x)^2 + (a/2)^2} = \sqrt{x^2 + a^2/4}$$

The unit vector is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{-x\hat{i} + a/2\hat{j}}{\sqrt{x^2 + a^2/4}}$$

The integral can now be written as

$$\vec{E} = k\lambda \int_{x=-L/2}^{x=L/2} \left(\frac{dx}{\sqrt{x^2 + a^2/4}^2} \frac{-x\hat{i} + a/2\hat{j}}{\sqrt{x^2 + a^2/4}} \right) = k\lambda \int_{x=-L/2}^{x=L/2} \left(\frac{(-x\hat{i} + a/2\hat{j})dx}{(x^2 + a^2/4)^{3/2}} \right)$$

† Problem from Univesity Physics by Ling, Sanny and Moebs (OpenStax)

This is really two integrals. One that involves the $\hat{\imath}$ term and one that involves the $\hat{\jmath}$ term.

$$\vec{E} = k\lambda \int_{x=-L/2}^{x=L/2} \left(\frac{(-x\hat{\imath})dx}{(x^2 + a^2/4)^{3/2}} \right) + k\lambda \int_{x=-L/2}^{x=L/2} \left(\frac{(a/2\hat{\jmath})dx}{(x^2 + a^2/4)^{3/2}} \right)$$

The $\hat{\imath}$ term equals zero since the point of interest is centered over the wire. Any field on the right-hand side will cancel out the left-hand side. (If you want to prove this to yourself mathematically, you can do a u-substitution with $u = x^2 + a^2/4$.) The $\hat{\jmath}$ term does not cancel because every location along the wire is contributing an upward electric field. This integral is solved using a tangent substitution. The result of the integral gives

$$\vec{E} = k\lambda(a/2)\hat{\jmath} \int_{x=-L/2}^{x=L/2} \left(\frac{dx}{\sqrt{x^2 + a^2/4}^{3/2}} \right) = k\lambda(a/2)\hat{\jmath} \left(\frac{x}{(a/2)^2(x^2 + a^2/4)^{1/2}} \right|_{-L/2}^{L/2}$$

$$\vec{E} = k\lambda(2/a)\hat{\jmath} \left(\frac{L/2}{((L/2)^2 + a^2/4)^{1/2}} - \frac{-L/2}{((-L/2)^2 + a^2/4)^{1/2}} \right)$$

$$\vec{E} = k\lambda(2/a)\hat{\jmath} \left(\frac{2(L/2)}{(L^2/4 + a^2/4)^{1/2}} \right) = k\lambda(2/a)\hat{\jmath} \left(\frac{L}{((L^2 + a^2)/4)^{1/2}} \right)$$

$$\vec{E} = k\lambda(2/a)\hat{\jmath} \left(\frac{L}{(1/2)(L^2 + a^2)^{1/2}} \right)$$

$$\vec{E} = \frac{4k\lambda L}{a(L^2 + a^2)^{1/2}}\hat{\jmath}$$

Now $\lambda = q/L$ and $k = \frac{1}{4\pi\epsilon_0}$, then

$$\vec{E} = \frac{4(1/4\pi\epsilon_0)(q/L)L}{a(L^2 + a^2)^{1/2}}\hat{\jmath}$$

$$\vec{E} = \frac{q}{\pi\epsilon_0 a(L^2 + a^2)^{1/2}}\hat{\jmath}$$

Now we can move on to the next case.

What is the electric field at P_2?

In this case, the vector r goes from a location on the wire a distance, x, from the origin to a point that is $L/2 + a$ from the origin in the x-direction. Therefore,

$$\vec{r} = (L/2 + a)\hat{\imath} - x\hat{\imath} = (L/2 + a - x)\hat{\imath}$$

The magnitude is

$$r = L/2 + a - x$$

The unit vector is

$$\hat{r} = \hat{\imath}$$

The integral can now be written as

$$\vec{E} = k\lambda \int_{x=-L/2}^{x=L/2} \left(\frac{dx}{(L/2 + a - x)^2}\hat{\imath} \right)$$

Do a u-substitution with

$$u = L/2 + a - x$$

$$du = -dx$$

The lower limit becomes

$$u_0 = L/2 + a - (-L/2) = L + a$$

The upper limit becomes

$$u_f = L/2 + a - (L/2) = a$$

The transformed integral is now

$$\vec{E} = k\lambda \int_{u=L+a}^{u=a} \left(\frac{-du}{u^2} \hat{i} \right)$$

$$\vec{E} = -k\lambda\hat{i} \left(\frac{-1}{u} \bigg|_{L+a}^{a} = k\lambda\hat{i} \left(\frac{1}{a} - \frac{1}{L+a} \right) \right.$$

Now $\lambda = q/L$ and $k = \frac{1}{4\pi\epsilon_0}$, then

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{L} \hat{i} \left(\frac{1}{a} - \frac{1}{L+a} \right)$$

or to match the solution in the textbook

$$\vec{E} = \frac{-q}{4\pi\epsilon_0 L} \hat{i} \left(\frac{1}{L+a} - \frac{1}{a} \right)$$