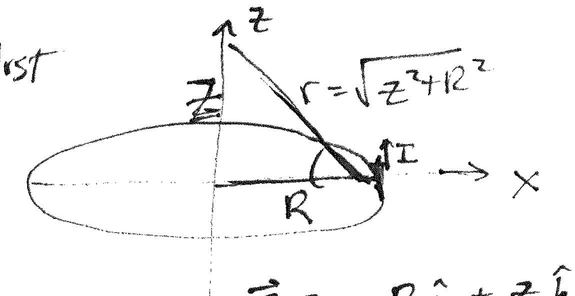
Ch.12 Prob. 75

a) Find the magnetic field in the middle of the Helmholtz coil

First find the magnetic field at location z, where $z=0$ is half way between the coils.

Let's do the field above 1 loop first



$$\vec{B} = \int \frac{I\mu_0}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

$$\vec{r} = -R\hat{i} + z\hat{k}$$

$$\hat{r} = \frac{-R\hat{i} + z\hat{k}}{\sqrt{z^2+R^2}}$$

The cross product gives

$$d\vec{l} \times \hat{r} = dl\,\hat{j} \times \left(\frac{-R\hat{i}+z\hat{k}}{\sqrt{z^2+R^2}}\right)$$

$$= \frac{R\,dl(-\hat{j}\times\hat{i})}{\sqrt{z^2+R^2}} + \frac{z\,dl(\hat{j}\times\hat{k})}{\sqrt{z^2+R^2}}$$

$$d\vec{l} \times \hat{r} = \frac{R\,dl\,\hat{k}}{\sqrt{z^2+R^2}} + \frac{z\,dl\,\hat{i}}{\sqrt{z^2+R^2}}$$

As you integrate around the circle the horizontal component of the B field cancels out.

∴ only the B field in the $\hat{k}$ direction will add together.

$$\text{so } \vec{B} = \int \frac{\mu_0 I}{4\pi}\left(\frac{R\,dl\,\hat{k}}{\sqrt{z^2+R^2}}\right)\frac{1}{\sqrt{z^2+R^2}^{\,2}} = \int \frac{\mu_0 I R \hat{k}\,dl}{4\pi(z^2+R^2)^{3/2}}$$

since I, R, and z are constant as you integrate around the loop you get

$$\vec{B} = \frac{\mu_0 I R \hat{k}}{4\pi(z^2+R^2)^{3/2}}\int dl$$

But the distance integrated around the loop equals the circumference of the loop

$$\therefore \quad \vec{B} = \frac{\mu_0 I R \hat{k}}{4\pi(z^2+R^2)^{3/2}}(2\pi R) = \frac{\mu_0 I R^2 \hat{k}}{2(z^2+R^2)^{3/2}}$$

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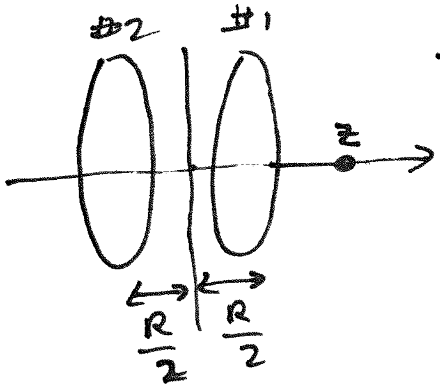
$$\vec{B} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}} \hat{k}$$

where z is how far the point of interest is from the center of the loop (Assuming the loop is at the origin)



for loop #1 the distance to the loop is $z - \frac{R}{2}$ (center of the loop is at $z = R/2$)

$$\therefore \vec{B}_1 = \frac{\mu_0 I R^2}{2\left[\left(z-\frac{R}{2}\right)^2 + R^2\right]^{3/2}}$$

for loop #2 the distance to the loop is $z + \frac{R}{2}$ (center of the loop is at $z = -R/2$)

$$\therefore \vec{B}_2 = \frac{\mu_0 I R^2}{2\left[\left(z+\frac{R}{2}\right)^2 + R^2\right]^{3/2}}$$

(a) Find B at any point on z-axis

The total B-field at location z is then

$$\vec{B} = \frac{\mu_0 I R^2}{2\left[z^2 - Rz + \frac{R^2}{4} + R^2\right]^{3/2}} + \frac{\mu_0 I R^2}{2\left[z^2 + Rz + \frac{R^2}{4} + R^2\right]^{3/2}}$$

$$\vec{B} = \frac{\mu_0 I R^2}{2}\left[\frac{1}{\left(z^2 - Rz + \frac{5}{4}R^2\right)^{3/2}} + \frac{1}{\left(z^2 + Rz + \frac{5}{4}R^2\right)^{3/2}}\right] \quad \boxed{\text{Eq. 1}}$$

$$\cancel{= \frac{\mu_0 I R^2}{2}\left(\frac{\left(z^2+Rz+5/4R^2\right)^{3/2} + \left(z^2-Rz+5/4R^2\right)^{3/2}}{\left(z^2-Rz+5/4R^2\right)^{3/2}\left(z^2+Rz+5/4R^2\right)^{3/2}}\right)}$$

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(b) show $\frac{dB}{dz}$ + $\frac{d^2B}{dz^2}$ are both 0 at $z=0$.

Now $$\frac{dB}{dz} = \frac{d}{dz}\left(\frac{\mu_0 I R^2}{2}\left[\frac{1}{(z^2-Rz+\frac{5}{4}R^2)^{3/2}} + \frac{1}{(z^2+Rz+\frac{5}{4}R^2)^{3/2}}\right]\right)$$

$$\frac{dB}{dz} = \frac{\mu_0 I R^2}{2}\left[\frac{-3/2\,(2z-R)}{(z^2-Rz+\frac{5}{4}R^2)^{5/2}} + \frac{-3/2\,(2z+R)}{(z^2+Rz+\frac{5}{4}R^2)^{5/2}}\right] \boxed{Eq.2}$$

and $$\frac{d^2B}{dz^2} = \frac{\mu_0 I R^2}{2}\left[\frac{-3/2\,(2)}{(z^2-Rz+\frac{5}{4}R^2)^{5/2}} + \frac{-3/2\,(-5/2)(2z-R)(2z-R)}{(z^2-Rz+\frac{5}{4}R^2)^{7/2}}\right.$$
$$\left. + \frac{-3/2\,(2)}{(z^2+Rz+\frac{5}{4}R^2)^{5/2}} + \frac{-3/2\,(-5/2)(2z+R)(2z+R)}{(z^2+Rz+\frac{5}{4}R^2)^{7/2}}\right]$$

$\boxed{Eq.3}$

At $z=0$ from $\boxed{Eq.1}$

$$B = \frac{\mu_0 I R^2}{2}\left[\frac{1}{(0^2-R(0)+\frac{5}{4}R^2)^{3/2}} + \frac{1}{(0^2+R(0)+\frac{5}{4}R^2)^{3/2}}\right]$$

$$= \frac{\mu_0 I R^2}{2}\left[\frac{1}{(\frac{5}{4}R^2)^{3/2}} + \frac{1}{(\frac{5}{4}R^2)^{3/2}}\right]$$

$$= \mu_0 I R^2\left[\frac{1}{\sqrt{(\frac{5}{4}R^2)^3}}\right] = \mu_0 I R^2\left[\frac{1}{\sqrt{\frac{125}{64}R^6}}\right]$$

$$B = \mu_0 I R^2\left[\frac{1}{\frac{\sqrt{125}}{8}R^3}\right] \Rightarrow \boxed{B = \frac{8\mu_0 I}{\sqrt{125}\,R}}$$

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At $z=0$ from $\boxed{Eq.2}$

$$\frac{dB}{dz} = \frac{\mu_0 I R^2}{2}\left[\frac{-3/2(2(0)-R)}{(0^2+R(0)+\frac{5}{4}R^2)^{5/2}} + \frac{-3/2(2(0)+R)}{(0^2+R(0)+\frac{5}{4}R^2)^{5/2}}\right]$$

$$= \frac{\mu_0 I R^2}{2}\left[\frac{3R}{(\frac{5}{4}R^2)^{5/2}} - \frac{3R}{(5/4R^2)^{5/2}}\right] = 0$$

At $z=0$ from $\boxed{Eq.3}$

$$\frac{d^2B}{dz^2} = \frac{\mu_0 I R^2}{2}\left[\frac{-3}{(0^2-R(0)+\frac{5}{4}R^2)^{5/2}} + \frac{\frac{15}{4}(2(0)-R)(2(0)-R)}{(0^2-R(0)+\frac{5}{4}R^2)^{7/2}}\right.$$

$$\left. + \frac{-3}{(0^2+R(0)+\frac{5}{4}R^2)^{5/2}} + \frac{\frac{15}{4}(2(0)+R)(2(0)+R)}{(0^2+R(0)+\frac{5}{4}R^2)^{7/2}}\right]$$

$$= \frac{\mu_0 I R^2}{2}\left[\frac{-3}{(\frac{5}{4}R^2)^{5/2}} + \frac{\frac{15}{4}R^2}{(5/4R^2)^{7/2}} - \frac{3}{(\frac{5}{4}R^2)^{5/2}} + \frac{\frac{15}{4}R^2}{(\frac{5}{4}R^2)^{7/2}}\right]$$

$$= \mu_0 I R^2\left[\frac{-3}{(5/4R^2)^{5/2}} + \frac{15R^2}{4(5/4R^2)^{7/2}}\right]$$

$$= \mu_0 I R^2\left[\frac{-3}{(\frac{5}{4}R^2)^{5/2}} + \frac{15R^2}{4(\frac{5}{4}R^2)^{5/2}\cdot(\frac{5}{4}R^2)^{2/2}}\right]$$

$$= \frac{\mu_0 I R^2}{(\frac{5}{4}R^2)^{5/2}}\left[-3 + \frac{15R^2}{4(\frac{5}{4}R^2)}\right] = \frac{\mu_0 I R^2}{(5/4R^2)^{5/2}}\left[-3 + \frac{15}{5}\right]$$

$$\therefore \frac{d^2B}{dz^2} = 0$$