Chapter 6 Problem 85 †

Given

$y = ax^2 - bx$
$a = 2\ m^{-1}$
$b = 4$
$\vec{F} = cxy\hat{\imath} + d\hat{\jmath}$
$c = 10\ N/m^2$
$d = 15\ N$

Solution

Find the work when going from $x = 3\ m$ to $x = 6\ m$.

Since we are working in 2 dimensions, the value of the differential displacements is $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$. For work the dot product becomes

$$W = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} (F_x dx + F_y dy) = \int_{r_1}^{r_2} (cxy\ dx + d\ dy) \tag{1}$$

Substitute in for y with the relationship $y = ax^2 - bx$ and for dy the relationship $dy = (2ax - b)dx$. This relationship comes from taking the derivative of the function for y.

$$\frac{dy}{dx} = \frac{d(ax^2 - bx)}{dx} = 2ax - b$$

Then multiply both sides by dx to get this relationship. Now substitute for y and dy in the work equation (1).

$$W = \int_{r_1}^{r_2} (cxy\ dx + d\ dy) = \int_{x=0}^{x=3} \big(cx(ax^2 - bx)\ dx + d(2ax - b)\ dx\big) = \int_{x=0}^{x=3} \big(cax^3 - cbx^2 + 2dax - db\big)\ dx$$

Notice that after this substitution the integral only depends on x and, therefore, the limits of integration only depend on x.
Now perform the integration with respect to x.

$$W = \left| \frac{cax^4}{4} - \frac{cbx^3}{3} + \frac{2dax^2}{2} - dbx \right|_{x=0}^{x=3}$$

Substituting in the values for a, b, c, and d and solving gives

$$W = \left(\frac{(10)(2)x^4}{4} - \frac{(10)(4)x^3}{3} + \frac{2(15)(2)x^2}{2} - (15)(4)x \right|_{x=0}^{x=3}$$

$$W = \left(5x^4 - \frac{40x^3}{3} + 30x^2 - 60x \right|_{x=0}^{x=3}$$

$$W = \left(5(3)^4 - \frac{40(3)^3}{3} + 30(3)^2 - 60(3) \right) - \left(5(0)^4 - \frac{40(0)^3}{3} + 30(0)^2 - 60(0) \right)$$

$$W = (405 - 360 + 270 - 180) - 0 = 135\ J$$

†Problem from Essential University Physics, Wolfson