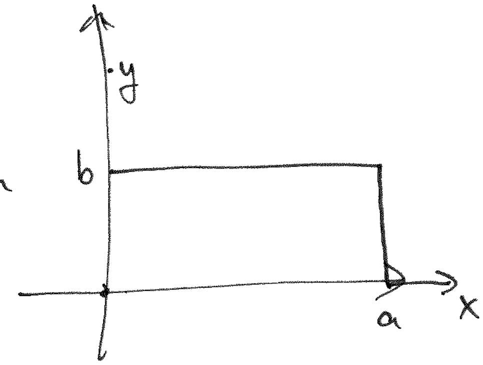
Chapter 9

Problem 70

$\rho(x,y) = \rho_0 x$

Assume the length, a, is in the x-direction



$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm$

Now density is the connection between volume and mass.

$M = \rho \cdot V$ and $dm = \rho \cdot dV$

In cartesian coordinates

$dV = dx\, dy\, dz$

Assuming a plate with constant properties in the y + z directions

Then $dV = dx(b)(t)$ where b is the width of the plate and t is the thickness of the plate

Now the center of mass in the x-direction is

$$X_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^a x\, \rho \cdot b \cdot t \cdot dx$$

$$= \frac{1}{M}\int_0^a \rho_0 \cdot x \cdot x \cdot b \cdot t \cdot dx = \frac{\rho_0 \cdot b \cdot t}{M}\int_0^a x^2 dx = \frac{\rho_0 b t}{M}\frac{x^3}{3}\Big|_0^a$$

$$\boxed{X_{cm} = \frac{\rho_0 b t}{3M} a^3}$$

The total mass of the plate is

$$M = \int dm = \int_0^a \rho \cdot b \cdot t \cdot dx = \int_0^a \rho_0 \cdot x \cdot b \cdot t \cdot dx$$

$$= \rho_0 \cdot b \cdot t \int_0^a x\, dx = \frac{\rho_0 b t}{2} x^2\Big|_0^a = \frac{\rho_0 b t}{2} a^2$$

Substitute into the previous result gives

$$X_{cm} = \frac{\rho_0 b t\, a^3}{3\left(\frac{\rho_0 b t}{2} a^2\right)} = \frac{2\rho_0 b t a^3}{3\rho_0 b t a^2} = \boxed{\frac{2}{3} a = X_{cm}}$$

In the y-direction there is no change in the density. Therefore the center of mass will be half-way between the edges at ~~x~~ $y=0$ and $y=b$

$$\therefore \boxed{Y_{cm} = b/2}$$